Nonequilibrium process in the σ model and chemical relaxation time in a homogeneous pionic gas

Mitsuru Ishii [*]

Yukawa Institute for Theoretical Physics, Kyoto, 606-01, Japan

Abstract

In a homogeneous pionic gas system, chemical nonequilibrium process is considered to understand its effect in the expansion processes that are realized immediately after heavy ion collisions. The chemical relaxation time is calculated by incorporating the $\pi+\pi \leftrightarrow \pi+\pi+\pi+\pi$ reaction, which is given in the second order of perturbation in the σ model. The $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction is assumed to be less frequent than the $\pi + \pi \leftrightarrow \pi + \pi$ scattering that is expected to establish the local equilibrium, and hydrodynamical equation is solved for various initial conditions. It is shown that the relaxation time is of the order of 100fm and does not have a significant effect on the expansion process, which implies that the pion number freezeout takes place at an early stage of the expansion.

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[*]e-mail: ishii@jpnyitp.bitnet

1 Introduction

It is widely believed that Quantum Chromo Dynamics (QCD) is the underlying theory of strong interaction and that QCD exhibits confinement-deconfinement phase transition at high temperature (≈ 200 MeV[1].) Since understanding the property of this phase transition is crucial for hadron physics, many efforts have been made both experimentally and theoretically [2]. In particular, heavy ion reaction is expected to be a useful method to establish high temperature with free quarks and gluons and to investigate the process of hadronization.

Heavy ion collision is considered to undergo several steps shown in Fig.1.:

1. Two highly Lorentz contracted nuclei collide with each other.

2. They pass through each other and a hot system is formed in between, where quarks and gluons are excited[3].

3. Frequent collisions and reactions among the excited elements lead to establishing local equilibrium. Therefore this state can be treated by hydrodynamics[4] [5].

4. As the system expands, cooling and subsequent phase transition process proceeds[2].

5. A lot of pions (and other light hadrons) are created under the process of the phase transition[6].

6. When the phase transition terminates, there exists hadronic gas which can also be treated by hydrodynamics. Since the system is hot, the collisions and reactions between the hadrons are frequent and the total number of hadrons is not conserved.

7. As the cooling goes on, the reaction to change the total number of hadrons becomes unlikely. At this stage, the multiplicity of pion (and other light hadrons) is fixed (freezeout of number)[7].

8. When the expansion proceeds further and the mean distance between pions is equivalent to the average variation of macroscopic quantity such as temperature, energy density and pressure, the hydrodynamical picture is not a good one any longer. Instead, the system can be considered to be composed of free pions with-

out any interaction. This stage is called the freezeout of temperature.

$$\boxed{\text{Fig.1}}$$

Therefore, the multiplicity of pion is determined at stage 7 not at the time when the phase transition terminates. Also, the energy distribution of pions is given by the thermal distribution at stage 8. In other words, hadronic final state of heavy ion collision does not directly reflect the property of the confinement-deconfinement phase transition

Due to this feature, it has been proposed to use electromagnetic and weak probe[8], which is free from the hadronic final state interaction, to investigate the hadronic property of the phase transition. However, if we can understand the process of the final state interaction, it is possible to extract the information on the phase transition by hadronic probes. In particular, pion multiplicity can be used to obtain the pion number and the entropy immediately after the phase transition: if one knows the elementary chemical process to change the pion multiplicity such as $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$, one can see how the the pion multiplicity varies through the cooling process, and from the information on the pion multiplicity obtained in heavy ion reactions one can track the process backward up to the time when the phase transition is over.

Whether the chemical reaction largely affects the cooling process or not depends on the balance between the cooling by the expansion and the speed of the chemical reaction. If the typical time scale of the chemical reaction is longer enough than that of the expansion, the freezeout of pion multiplicity takes place at an early stage of the expansion and one can identify the final pion multiplicity with the pion multiplicity immediately after the phase transition in the first approximation. In this case we can estimate the entropy density immediately after the phase transition (s) by

$$s = 4n, \quad n : \text{observed pion multiplicity}, \tag{1}$$

which is obtained for massless classical particle systems with the particle numbers conserved[9]. If the former is shorter than the latter, the chemical reaction remains dominant through the expansion. In this case, the final pion multiplicity is quite different from the initial

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one at the end of the phase transition, and Eq.(1) can not be used to obtain the entropy density. Instead, one must track the irreversible process backward, estimate the pion multiplicity at the end of the phase transition and then use Eq.(1).

In this paper, we aim to obtain the speed of the chemical reaction $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ based on the σ model. In this model, the interaction in the leading order of perturbation just gives $\pi + \pi \leftrightarrow \pi + \pi$ scattering, which is expected to establish local equilibrium in a pionic gas. In the second order of the perturbation, $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction is derived. If the latter reaction is less frequent than the former one, we can resort to hydrodynamical equation to track the change of pion number multiplicity. In section 2, the evaluation of the frequency of these two reaction is made and the criterion to establish local equilibrium is given. Then, we present hydrodynamical equation in the classical approximation. In section 3, the hydrodynamical equation is solved numerically in a homogeneous pion gas for various initial temperatures and chemical potentials. The typical time scale of chemical reaction (relaxation time) is obtained and compared to the expansion rate. The last section 4 is devoted to summary and discussing open problems.

2 Elementary chemical reaction in the σ-model and the hydrodynamical equation

We take σ-model to describe the $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction. The interaction part of the Lagrangian is given by

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}(\sigma^2 + \pi^2 - f_\pi{}^2)^2, \tag{2}$$

where $\pi = \pi^1 \tau_1 + \pi^2 \tau_2 + \pi^3 \tau_3$ and f_π is the pion decay constant. After shifting the σ field as $\sigma \rightarrow \sigma + f_\pi$, one obtains $-\lambda/4!(\pi^2 + 2f_\pi\sigma + \sigma^2)^2$ as the potential term. This term gives $\pi + \pi \leftrightarrow \pi + \pi$ scattering depicted in Fig.2(a). Summing all the diagrams in Fig.2(a) at threshold $p_i = (m, 0, 0, 0)$, $(i = 1, 2, 3, 4)$ gives

$$-\lambda + (\frac{\lambda}{3})^2 \frac{f_\pi{}^2}{m_\sigma{}^2} + (\frac{\lambda}{3})^2 \frac{f_\pi{}^2}{m_\sigma{}^2} + (\frac{\lambda}{3})^2 \frac{f_\pi{}^2}{m_\sigma{}^2 - (2m)^2}, \tag{3}$$

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where m_σ (m) is the σ (π) meson mass. Since $m_\sigma{}^2 = \lambda f_\pi{}^2/3$, we get

$$-\frac{\lambda}{3} + \frac{\lambda}{3}\frac{1}{1-(\frac{2m}{m_\sigma})^2} \tag{4}$$

as an effective 4π coupling constant. For $m_\sigma \approx 600$ MeV, $m/m_\sigma \approx 1/4$ and the effective coupling constant is $\lambda/9$. For simplicity, we hereafter replace the sum of Fig.1(a) by just one direct 4π coupling with its coupling constant $\lambda/9$.

The $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction is given in the second order of perturbation (Fig.2(b)). In the second order, another diagram is possible for the $\pi+\pi \leftrightarrow \pi+\pi+\pi+\pi$ process (Fig.2(c)). This diagram, however, is not taken into account because the denominator of the propagator $p^2 - m^2$ is of the order of $(3m)^2 - m^2$ and the contribution of Fig.2(c) is $O(1/10)$ of Fig.2(b). All the possible diagrams of type (b) are shown in Fig.2(d). For simplicity, we replace Fig2.(d) by Fig2.(e), which means the effect of interference is not exactly taken into account.

$$\boxed{\text{Fig.2}}$$

In the next section, the hydrodynamical equation is solved assuming local equilibrium that is expected to be established by the $\pi + \pi \leftrightarrow \pi + \pi$ scattering. Since $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction drives the system away from equilibrium, $\pi + \pi \leftrightarrow \pi + \pi$ scattering must be frequent enough so that the disturbed nonequilibrium state can swiftly return to equilibrium state.

Once local equilibrium is established, local temperature $T(x)$ as well as local chemical potential $\mu(x)$ is defined. The $\pi + \pi \leftrightarrow \pi + \pi$ actually means $\pi^0+\pi^0 \leftrightarrow \pi^++\pi^-$ and $\pi^i+\pi^j \leftrightarrow \pi^i+\pi^j$ ($i,j = +,-,0$). The first one gives the equilibrium condition $2\mu^0(x) = \mu^+(x)+\mu^-(x)$. Since the charge conservation means $\mu^+(x) = \mu^-(x)$ in neutral pionic gas, we get $\mu^+(x) = \mu^-(x) = \mu^0(x)$ or $n^{\pi^+}(x) = n^{\pi^-}(x) = n^{\pi^0}(x)$ (n: pion density.) This means that all the equations are identical for π^0, π^+ and π^-.

In other words, all the calculation is made in the same way as in $\lambda\phi^4$ theory except for a numerical factor stemming from the isospin degrees of freedom, because the essential part of the σ model is only the $(\lambda/4!)\pi^4$ term.

As we have mentioned before, the $\pi+\pi \leftrightarrow \pi+\pi$ scattering should be more frequent than $\pi+\pi \leftrightarrow \pi+\pi+\pi+\pi$ reaction in order to have

the system locally equilibrated. The frequency of $\pi + \pi \leftrightarrow \pi + \pi$ and $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ process are calculated in Appendix A:

$$
\begin{aligned}
I^{\pi+\pi\leftrightarrow\pi+\pi} &\equiv \int \frac{d\mathbf{p}_1}{(2\pi)^3 2p_1{}^0} \frac{d\mathbf{p}_2}{(2\pi)^3 2p_2{}^0} \frac{d\mathbf{p}_3}{(2\pi)^3 2p_3{}^0} \frac{d\mathbf{p}_4}{(2\pi)^3 2p_4{}^0} \\
&\quad \times (\frac{\lambda}{9})^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) e^{-\beta(p_1+p_2)^0+2\beta\mu} \\
&= (\frac{\lambda}{9})^2 \frac{1}{32 \ (2\pi)^5} e^{2\beta\mu} \int_{2m}^{\infty} dP_0 e^{-\beta P_0} \\
&\quad \times \int_0^{\sqrt{P_0^2-4m^2}} d|\mathbf{p}||\mathbf{p}|^2 \sqrt{1 - \frac{4m^2}{(P_0^2 - \mathbf{p}^2)}}
\end{aligned}
\tag{5}
$$

$$
I = \frac{575}{243} \ 3 \ (\frac{\lambda}{9})^4 \frac{1}{128(2\pi)^9} e^{-4\beta} m^4 (e^{2\beta\mu} - e^{4\beta\mu}) \int_0^{\infty} x^2(x+8)g(x)e^{-\beta x} dx
\tag{6}
$$

where $\beta = 1/T$, T is the temperature, μ the chemical potential, m the pion mass, P_0 the total energy of the colliding pions, $\mathbf{p}$ their total spatial momentum, $P_0/m = 4 + x$ and $g(x)$ a function independent of temperature and chemical potential whose definition is given in Appendix A. The factor $\frac{575}{243}$ comes from the isospin degrees of freedom (see Appendix B.) Since no further analytical calculation is possible, we carried out the integrals numerically for $m\beta \approx 1$ and $\mu/m \approx 1$ to get a rough estimate for $I^{\pi+\pi\leftrightarrow\pi+\pi}$ and I. It was found that $\lambda \ll 1000$ is required for $I^{\pi+\pi\leftrightarrow\pi+\pi} \gg I$. Since $\lambda \approx 100$ is taken for the ordinary σ-model, local equilibrium is established in the present case.[1]

Under local equilibrium, relativistic hydrodynamical equation[10] is given as :

$$
\partial_\mu N^{i\mu} = 2I \tag{7}
$$
$$
\partial_\mu T^{i\mu\nu} = 0 \tag{8}
$$

for each species of pion, where $N^{i\mu}(i = +, -, 0)$ is pion number current and $T^{i\mu\nu}$ is the energy momentum tensor for π^+, π^- and π^0 respectively. As is mentioned above, these fundamental equations take the

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[1] Since $\pi + \pi \leftrightarrow \pi + \pi$ is $O(\lambda^2)$ and $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ is $O(\lambda^4)$, it may seem that $\lambda \ll 1$ is required for local equilibrium. However, numerical factors coming from the phase space integral such as $1/(2\pi)^3$ for each external line and $e^{-4\beta}$ in Eq.(6) suppress the frequency of the $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ process. Thus $\lambda \approx 100$ satisfies the criterion for local equilibrium even if λ is bigger than 1.

same form for all the species of pions. Hence, we hereafter suppress the indices $i = +, -, 0$. Eq.(7) describes the change of the total pion number and is called rate equation, whose right hand side $2I$ gives the rate of particle production and annihilation. The factor 2 in the right hand side comes from the fact that two pions are created for one elementary reaction in Fig.2(b). Eq.(8) is the energy-momentum conservation identity.

N^μ and $T^{\mu\nu}$ are expressed in terms of the pion distribution function f:

$$N^\mu = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p^\mu}{p^0} f \qquad (9)$$

$$T^{\mu\nu} = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p^\mu p^\nu}{p^0} f. \qquad (10)$$

3 Numerical solution for the hydrodynamical equation and chemical relaxation time in a homogeneous pionic gas

In this section, we consider homogeneous pionic gases to obtain the typical time scale of the chemical reaction $\pi + \pi \leftrightarrow \pi + \pi$. In a homogeneous gas, the hydrodynamical equation Eqs.(7)(8) is drastically simplified to[4]

$$\frac{dn}{dt} = 2I \qquad (11)$$

$$\frac{d\epsilon}{dt} = 0, \qquad (12)$$

where $n = N^0 (\epsilon = T^{00})$ is the pion density (energy density). To express N^μ and $T^{\mu\nu}$ in terms of the local temperature T and local chemical potential μ, we need to have the explicit form of the distribution function f. In this paper, we take classical distribution

$$f(p, x, t) = \exp\{-\beta(x, t)(\sqrt{\mathbf{p}^2 + m^2} - \mu(x, t))\}. \qquad (13)$$

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Since we investigate homogeneous pionic gas, there is no x dependence in any macroscopic quantity and T and μ are the functions of time t alone. Then, we get

$$n(t) = \frac{1}{2\pi^2} e^{\mu/T} T m^2 K_2(\frac{m}{T}) \tag{14}$$

$$\epsilon(t) = \frac{1}{2\pi^2} e^{\mu/T} m^2 \{3T^2 K_2(\frac{m}{T}) + TmK_1(\frac{m}{T})\} \tag{15}$$

Combining Eqs.(11)(12) with Eqs.(14)(15), one gets a set of differential equation for $T(t)$ and $\mu(t)$. The energy conservation law Eq.(12) means $\epsilon(t) = \epsilon(t=0)$ giving the equation to determine $\mu(t)$ from $T(t)$ and initial conditions $T_0 = T(0), \mu_0 = \mu(0)$:

$$\mu = \frac{\mu_0}{T_0} T + T \ln\{(\frac{T_0}{T})^2 \frac{3K_2(m/T_0) + m/T_0 K_1(m/T_0)}{3K_2(m/T) + m/TK_1(m/T)}\}, \tag{16}$$

with the abbreviation $\mu = \mu(t), T = T(t)$. Also from $d\epsilon/dt = 0$, we can express $d\mu/dt$ in terms of $T(t), \mu(t), dT/dt, T_0$ and μ_0:

$$\dot{\mu} = (\frac{\mu}{T} - A)\dot{T} \tag{17}$$

$$A \equiv \frac{12K_2(m/T) + 5(m/T)K_1(m/T) + (m^2/T^2)K_0(m/T)}{3K_2(m/T) + (m/T)K_1(m/T)}, \tag{18}$$

where $\dot{\mu}$ $(\dot{T})$ means $d\mu/dt$ (dT/dt). By eliminating μ and $\dot{\mu}$ and re-defining all the quantities to be dimensionless, i.e. $T/m \to T \equiv 1/\beta, \mu/m \to \mu, \lambda^4 mt \to t$, we can rewrite the hydrodynamical equation as

$$\{(3 - A)K_2(\beta) + \beta K_1(\beta)\}\dot{T} = 2I/\lambda^4. \tag{19}$$

Since I is proportional to λ^4 (see Eq.(6)), the λ^4 term in the redefinition of t makes the equation independent of λ.

By solving this differential equation, one gets the t dependence of T and from Eq.(16) $\mu(t)$ is also obtained. The results of numerical calculation are shown in Figs.3,4,5,6 for initial conditions $(T_0, \mu_0) = (0.5, -1), (1.5, -1), (0.5, 1), (1.5, 1)$ respectively.

$$\boxed{\text{Figs.3,4,5,6}}$$

From these figures, one can see that the relaxation is faster for hotter and denser (larger μ_0) systems, which is reasonable considering that the chemical reaction is more frequent for hotter and denser systems. Since the relaxation is not exactly exponential, we define relaxation time by $\mu(t_{\text{relax}}) = \mu_0/e$. The relaxation times with $\lambda = 120$ are shown in Fig.7 for various initial conditions $-1 < \mu_0 < 1, 0.5 < T_0 < 1.5$. For a dense and hot system (e.g. $\mu_0 = 1$ and $T_0 = 1.5$), t_{relax} is of the order of 100fm. This time scale is expected to be larger than the typical time scale of expansion. Thus, we have found that the $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction does not play an important role in an expanding pionic gas system. In other words, the pion number freezes out at an early stage of the final state of a heavy ion reaction. This implies that the hadronic probe is a good one to extract information on the phase transition and that $s = 4n$ is an appropriate one to evaluate the entropy density immediately after the phase transition,

Fig.7

4 Summary and open problems

We have investigated the effect of $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ chemical reaction in the framework of hydrodynamics. To understand the typical time scale of the $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction, we calculated the relaxation times in homogeneous pionic gas systems. It turned out that the relaxation time is of the order of 100fm at shortest and is longer than typical time scale of the expansion of the pionic gas.

Although we found that the chemical $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction does not play an important role within the framework of the σ-model, there still are a lot of things to be done. The first one is to take into account the effect of the baryon in the central region, which has been neglected so far. In the real case, thermally excited $N\bar{N}$ pairs exist in the central region. Since the πN interaction is much larger than the $\pi\pi$ interaction, the processes such as $N + \pi \leftrightarrow N + \pi + \pi$ might be significant even if the number of the excited $N\bar{N}$ pairs is small. Furthermore, if there is a stopping in heavy ion collisions and there exists baryon number coming from the colliding nuclei, there would be a big change of pion multiplicity.

Secondly, to make reliable quantitative discussion on the role of the $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction, the calculation must be made without

classical approximation incorporating the effect of interference. This is expected to make the collision less frequent and lead to a larger relaxation time, keeping the essential statement in the former section. The third thing is to take the medium effect into account. So far, we have fixed the parameters in the σ-model: the coupling constant λ and the pion mass m. It is known, however, that these quantities depend on temperature and chemical potential. The correction due to this effect might not be negligible.

Even if some of these effects lead to a smaller relaxation time and make the effect of the change of the pion number a little more important, we still do not expect them to change our results qualitatively because of the following reason. In an expanding system, the chemical potential at the end of the phase transition is taken to be 0, because there are strong interaction between QGP and pions and pion can change its multiplicity easily. Therefore, at an early stage of the expansion in which the system is hot enough, the chemical potential is close to 0 and the system is close to chemical equilibrium. So, $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction etc. can not play an important role. As the system expands, the chemical potential deviates from 0. However, the system is not hot enough and the relaxation time is too long to make the effect of $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction significant.

Acknowledgement

Appendix A

In this appendix, we show how to calculate the frequency of $\pi + \pi \leftrightarrow \pi + \pi$ scattering and $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ reaction. For simplicity, we neglect the isospin degrees of freedom, which will be taken into account in Appendix B.

Under classical approximation, the frequency of the process depicted in Fig.2(a) per unit volume and unit time is given by

$$
\begin{aligned}
I^{\pi+\pi \leftrightarrow \pi+\pi} &= \int \frac{d\mathbf{p}_1}{(2\pi)^3 2p_1{}^0} \frac{d\mathbf{p}_2}{(2\pi)^3 2p_2{}^0} \frac{d\mathbf{p}_3}{(2\pi)^3 2p_3{}^0} \frac{d\mathbf{p}_4}{(2\pi)^3 2p_4{}^0} \\
&\times (\frac{\lambda}{9})^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) e^{-\beta(p_1+p_2)^0 + 2\beta\mu} \quad \text{(A.1)}
\end{aligned}
$$

We first carry out the integral over $d\mathbf{p}_3$ and $d\mathbf{p}_4$. Rewriting the integral in an explicitly Lorentz invariant form by using

$$
\int \frac{d\mathbf{p}_3}{(2\pi)^3 2p_3{}^0} = \int dp_3 \, \theta(p_3^0) \, \delta(p_3^2 - m^2) \quad \text{(A.2)}
$$

and then changing the variables to $p = p_3 + p_4, q = \frac{p_3 - p_4}{2}$, we get

$$
\begin{aligned}
& \int \frac{d\mathbf{p}_3}{(2\pi)^3 2p_3{}^0} \frac{d\mathbf{p}_4}{(2\pi)^3 2p_4{}^0} \delta(p_3 + p_4 - p_1 - p_2) \\
&= \int dp_0 d\mathbf{p} \int_{-\frac{|\mathbf{p}|}{2}\sqrt{1-\frac{4m^2}{p^2}}}^{\frac{|\mathbf{p}|}{2}\sqrt{1-\frac{4m^2}{p^2}}} dq_0 \frac{1}{|\mathbf{q}||\mathbf{p}|} 2\pi^2 \int d|\mathbf{q}||\mathbf{q}|^2 \frac{1}{2} \delta(p - p_1 - p_2) \\
&\times \int d(\cos\theta)\delta(\cos\theta - \frac{p^0 q^0}{|\mathbf{p}||\mathbf{q}|})\delta(|\mathbf{q}| - \sqrt{q_0^2 + \frac{p^2}{4} - m^2}) \frac{1}{2\sqrt{q_0^2 + \frac{p^2}{4} - m^2}} \\
&= \frac{\pi}{2}\sqrt{1 - \frac{4m^2}{(p_1 + p_2)^2}}. \quad \text{(A.3)}
\end{aligned}
$$

The integral over $d\mathbf{p}_1$ and $d\mathbf{p}_2$ is made essentially in the same way. However, the factor $\sqrt{1 - \frac{4m^2}{(p_1+p_2)^2}}$ in the above equation now prevents full analytical calculation. The final result is

$$
\begin{aligned}
I^{\pi+\pi \leftrightarrow \pi+\pi} &= (\frac{\lambda}{9})^2 \frac{1}{32\,(2\pi)^5} e^{2\beta\mu} \int_{2m}^{\infty} dP'_0 e^{-\beta P'_0} \\
&\times \int_0^{\sqrt{P_0'^2 - 4m^2}} d|\mathbf{p}||\mathbf{p}|^2 \sqrt{1 - \frac{4m^2}{(P_0'^2 - \mathbf{p}^2)}}, \quad \text{(A.4)}
\end{aligned}
$$

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where $P'_0 = (p_1 + p_2)^0$ and $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$.

The collision frequency for Fig.2(b) is made almost in the same manner. All the possible diagram of type Fig2.(b) is given in Fig2.(d). We approximate Fig2(d) by Fig2.(e), which means that we have replaced the interference term by Fig.2(e). This approximation makes it possible to carry out the integral over the relative variables between p_1 and p_2 and between p_3 and p_4. The rate of the particle production of the processes $\pi + \pi \leftarrow \pi + \pi + \pi + \pi$ depicted in Fig.2(e)is written as

$$
\frac{1}{4!2!} \int \frac{d\mathbf{p}_1}{(2\pi)^3 2p_1^{\;0}} \frac{d\mathbf{p}_2}{(2\pi)^3 2p_2^{\;0}} \frac{d\mathbf{p}_3}{(2\pi)^3 2p_3^{\;0}} \frac{d\mathbf{p}_4}{(2\pi)^3 2p_4^{\;0}} \frac{d\mathbf{p}_5}{(2\pi)^3 2p_5^{\;0}} \frac{d\mathbf{p}_6}{(2\pi)^3 2p_6^{\;0}}
$$
$$
\times \quad e^{2\beta\mu} e^{-\beta(p_5 + p_6)^0} |6M|^2 (2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4 - p_5 - p_6), \qquad (A.5)
$$

where M is the matrix element of the Fig.2(b):

$$
M = (\frac{\lambda}{9})^2 \frac{1}{m^2 - (p_1 + p_2 - p_5)^2}. \qquad (A.6)
$$

The rate for the reverse reaction $\pi + \pi + \pi + \pi \leftarrow \pi + \pi$ is given by replacing $e^{2\beta\mu} e^{-\beta(p_5 + p_6)^0}$ by $e^{4\beta\mu} e^{-\beta(p_1 + p_2 + p_3 + p_4)^0}$. The difference from Eq.(A.1) is that the propagator in M hampers analytically carrying out the integral of $d(p_1 + p_2)$ and $d(p_3 + p_4)$ as is shown below.

Taking $p_1 + p_2 = p$, $\frac{p_1 - p_2}{2} = q$, $p_3 + p_4 = p'$, $\frac{p_3 - p_4}{2} = q'$ and carrying out the integral over dq and dq' as is done in Eq.(A.2), we get

$$
\frac{3}{4} e^{2\beta\mu} \frac{1}{(2\pi)^{18}} (\frac{\pi}{2})^2
$$
$$
\times \int_{p^2 > 4m^2} dp \sqrt{1 - \frac{4m^2}{p^2}} \int_{p'^2 > 4m^2} dp' \sqrt{1 - \frac{4m^2}{p'^2}} \int dp_5 dp_6
$$
$$
\times \delta(p_5^{\;2} - m^2) \delta(p_6^{\;2} - m^2) \theta(p_5^{\;0}) \theta(p_6^{\;0}) \delta(p + p' + p_5 + p_6)
$$
$$
\times e^{-\beta(p_5 + p_6)^0} \{ \frac{1}{m^2 - (p - p_5)^2} \}^2. \qquad (A.7)
$$

Then, we rewrite $p_5 + p_6 = p_t$, $\frac{p_5 - p_6}{2} = q_t$, $p + p' = P$, $\frac{p - p'}{2} = Q$ and define

$$
\begin{aligned}
P &= (P^0, 0, 0, |\mathbf{P}|) \\
Q &= (Q_0, |\mathbf{Q}| \sin\theta_1, 0, |\mathbf{Q}| \cos\theta_1) \\
q_t &= (q_t^{\;0}, |\mathbf{q}| \sin\theta_2 \cos\varphi, |\mathbf{q}| \sin\theta_2 \sin\varphi, |\mathbf{q}| \cos\theta_2). \qquad (A.8)
\end{aligned}
$$

p_t can be eliminated due to the total momentum conservation δ function. The other two δ functions are written as

$$\delta(p_5{}^2 - m^2)\,\delta(p_5{}^2 - m^2) = \delta(\frac{P^2}{4} + Q^2 - m^2)\,\delta(2P \cdot Q) \qquad \text{(A.9)}$$

and enable us to eliminate $\cos\theta_2$ and $|\mathbf{q}_t|$. The integration over $d\varphi$ is easily made and after some manipulation to make the expression convenient for Monte Carlo calculation, we get

$$
\begin{aligned}
I \;=\; & \frac{3}{128(2\pi)^9}(\frac{\lambda}{9})^4 e^{-4\beta} m^4 (e^{2\beta\mu} - e^{4\beta\mu}) \\
& \times \int_0^\infty x^2(x+8)g(x)e^{-\beta x}dx,
\end{aligned}
\qquad \text{(A.10)}
$$

$$
\begin{aligned}
g(x) \;=\; & \int_0^1 dx_2 \int_{-1}^1 dx_3 \int_0^1 dx_4 \int_{-1}^1 dz \int_{-1}^1 dy \\
& \times\; l^2 \sqrt{(1 - \frac{4}{\frac{P^2}{4} + Q^2 + P \cdot Q})(1 - \frac{P^2}{4} + Q^2 - P \cdot Q)}\,\frac{|\mathbf{P}|}{2}\sqrt{1 - \frac{4}{P^2}}\,\frac{a}{\sqrt{a^2 - b^2}} \\
& \times\; \theta(\frac{P^2}{4} + Q^2 - 4 - |P \cdot Q|)\, x_2\, x_4^2
\end{aligned}
\qquad \text{(A.11)}
$$

with dimensionless variables redefined by

$$
P = \frac{p_1 + p_2 + p_3 + p_4}{m} = \frac{p_5 + p_6}{m},\; Q = \frac{(p_1 + p_2 - p_3 - p_4)}{2m},\; q_t = \frac{(p_5 - p_6)}{2m},
$$

$$
P^0 = 4 + x,\; |\mathbf{P}| = \sqrt{P^{0\,2} - 16}\, x_2 = \sqrt{x(x+8)}\, x_2,\; Q_0 = \frac{x}{2}x_3,\; |\mathbf{Q}| = lx_4,
$$

$$
q_t{}^0 = \frac{|\mathbf{P}|}{2}\sqrt{1 - \frac{4}{P^2}}\, y,\; |\mathbf{q}_t| = \sqrt{\frac{P_0{}^2 - |\mathbf{P}|^2}{4} - 1 + (q_t{}^0)^2},
$$

$$
l = \sqrt{\frac{P^{0\,2} - 16}{4} + Q^{0\,2}},\; a = \frac{P^2}{4} - Q^2 + 2Q^0 q_t{}^0 - 2|\mathbf{Q}||\mathbf{q}_t|z_1 z_2,
$$

$$
b = 2|\mathbf{Q}||\mathbf{q}_t|\sqrt{1 - z_1^2}\sqrt{1 - z_2^2},\; z = z_1 = \cos\theta_1,\; z_2 = \cos\theta_2 = \frac{P^0 q_t{}^0}{|\mathbf{p}||\mathbf{q}_t|}|. \quad \text{(A.12)}
$$

The θ function comes from the constraints

$$p^2 = (\frac{P}{2} + Q)^2 > 4m^2 \quad \text{and} \quad p'^2 = (\frac{P}{2} - Q)^2 > 4m^2 \qquad \text{(A.13)}$$

The advantage of taking the classical distribution is that the thermal factor is given as $e^{-\beta P_0}$ and all the variable other than the total energy can be integrated regardless of temperature. The temperature independent part is written as $g(x)$ whose x (or P_0) dependence is shown in Fig.A.

$$\boxed{\text{Fig.A}}$$

Appendix B

In this appendix, we consider the effect of the isospin degrees of freedom. For simplicity, we take π^0,π^1,π^2 not π^0,π^+,π^- as three species of pions. One can easily confirm that the calculation based on π^0,π^+,π^- leads to the same conclusion.

The factor attached to four point vertices depends on whether the coupling concerns only one species of pions or not: $-\lambda$ for the $4\pi^0$ ($4\pi^1$ and $4\pi^2$) vertices and $-\lambda/3$ for the $2\pi^0$-$2\pi^1$ ($2\pi^1$-$2\pi^2$ and $2\pi^2$-$2\pi^0$) vertices (see Fig.B.1.)

$$\boxed{\text{Fig.B.1}}$$

First, let us take the $\pi^0 + \pi^0 \leftrightarrow 4\pi$ processes with every possible combination of 4π such as $\pi^+ + \pi^- + \pi^0 + \pi^0$. All the possible Feynman diagrams are shown in Fig.B.2. Since the three (two) diagrams in Fig.B.2(c) and Fig.B.2(c') (Fig.B.2(d)) have the same initial and final state, they must be added before taking the square and interfere with each other. Then we have to take possible combinations of allocating the external lines to the initial and final pions. As is mentioned in Chap.2, we make the approximation that all the kinds of phase space integrals give the same result. So, we just multiply the number of combinations to the simplest diagrams shown in Fig.B.2 and then put appropriate factors to avoid overcounting of the initial and final states. Now that phase space integrals are the same for all the diagrams, we just need to calculate the factors coming from the vertices ($-\lambda$ and $-\lambda/3$) and from the combination of allocating the external lines shown above.

$$\boxed{\text{Fig.B.2}}$$

These calculations are easily made and the diagrams in Fig.B.2

give $\frac{3}{4}\lambda^4 + \frac{1}{3}\lambda^4$ for $\pi^1 + \pi^1 \leftrightarrow 4\pi$. The calculations for $\pi^1 + \pi^1 \leftrightarrow 4\pi$ and $\pi^2 + \pi^2 \leftrightarrow 4\pi$ are made exactly in the same way, and one obtains $3(\frac{3}{4}\lambda^4 + \frac{1}{3}\lambda^4)$ as the sum of $\pi^i + \pi^i \leftrightarrow 4\pi$ $(i = 1, 2, 3)$ reactions.

Next, we need to consider those reactions from (to) two different species of pions to (from) all the possible four pions. As an example, we take $\pi^0 + \pi^1 \leftrightarrow 4\pi$. The possible processes are depicted in Fig.B.3. The calculations are essentially the same as before, and we get $\frac{56}{81}\lambda^4$ as the sum of these diagrams. The same results are obtained for the other two processes $\pi^1 + \pi^2 \leftrightarrow 4\pi$ and $\pi^2 + \pi^0 \leftrightarrow 4\pi$.

$$\boxed{\text{Fig.B.3}}$$

Thus all the contribution of isospin degrees of freedom is $3(\frac{3}{4} + \frac{1}{3} + \frac{56}{81})\lambda^4$, or the increase and decrease for each species of pion is characterized by $(\frac{3}{4} + \frac{1}{3} + \frac{56}{81})\lambda^4$. In other word, the effect of isospin degrees of freedom can be taken into account by replacing $\frac{3}{4}\lambda^4$, which is the factor in the case of $\pi^0 + \pi^0 \leftrightarrow \pi^0 + \pi^0 + \pi^0 + \pi^0$ given by Fig.B.2(a), by $(\frac{3}{4} + \frac{1}{3} + \frac{56}{81})\lambda^4$. The ratio $(\frac{3}{4} + \frac{1}{3} + \frac{56}{81})\lambda^4/\frac{3}{4}\lambda^4$ is the numerical factor $\frac{575}{243}$ in Eq.(6).

Figure captions

Fig.1: The schematic picture of heavy ion collision in t-z space.

Fig.2: The elementary process under consideration. (a):$\pi+\pi \leftrightarrow \pi+\pi$ scattering. (b):$\pi+\pi \leftrightarrow \pi+\pi+\pi+\pi$ process. (c):the other possible $\pi+\pi \leftrightarrow \pi+\pi+\pi+\pi$ process in the second order of perturbation in the σ model. This diagram is neglected. (d):All the possible detailed diagram of type (b). (e):Approximated diagram for (d) used in the numerical calculation. All the 4π vertices in (b),(c),(d),(e) are the effective coupling obtained from the sum of diagrams in (a).

Fig.3.4.5.6: The time dependence of (a):temperature T, (b):chemical potential μ, and (c):$\log|\mu|$ for initial conditions $(T_0, \mu_0) = (0.5, -1.0)$ (Fig.3),(1.5,-1) (Fig.4), (0.5,1) (Fig.5) and (1.5,1.0) (Fig.6). The temperature T, chemical potential μ, time t are all dimensionless ones defined in the text.

Fig.7: $\log_{10} t_{\text{relax}}$ for various initial conditions with $\lambda = 120$.

Fig.A.: x dependence of the temperature independent function $g(x)$.

Fig.B.1: Two kinds of 4π coupling in the σ model.

Fig.B.2: All the possible Feynman diagrams for $\pi^0 + \pi^0 \leftrightarrow 4\pi$

Fig.B.2: All the possible Feynman diagrams for $\pi^0 + \pi^1 \leftrightarrow 4\pi$

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